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A THEOREM CONCERNING CYCLIC DIRECTED GRAPHS WITH APPLICATIONS T--ETC(U)

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by
RANDALL R. WILLIE

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WITH APPLICATIONS TO NETWORK RELIABILITY

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Operations Research Center
University of California, Berkeley

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I am grateful to Professor Richard Barlow of University of California, Berkeley for bringing Reference [1] to my attention and helping me to understand some of the finer points of that paper. The concise proof of Lemma 1 was suggested by Ms. Jane Hagstrom of the Department of Industrial Engineering and Operations Research, University of California, Berkeley. Finally, Professor Richard Karp, also of University of California, Berkeley, provided several valuable suggestions regarding the proof of Lemma 2.

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ABSTRACT

In a recent paper, Satyanarayana and Prabhaker have presented a new topological formula for evaluating exact reliability of terminal-pair directed networks. Terms in the formula are associated in a one-to-one fashion with certain acyclic subgraphs of the network, cyclic subgraphs being of no importance. In their paper, however, the proof that cyclic subgraphs may be ignored seems to be incomplete. We consider an alternate proof of this fact.

A THEOREM CONCERNING CYCLIC DIRECTED GRAPHS
WITH APPLICATIONS TO NETWORK RELIABILITY

by

Randall R. Willie

In a recent incisive and original paper, Satyanarayana and Prabhaker [1] have suggested a new and efficient method for calculating exact system reliability for two terminal networks. The method applies to a directed graph G with a single source vertex s and a sink vertex t and with distinct initial and terminal vertices for each edge. Any edge ω may be working or failed, with probabilities p_ω and $1 - p_\omega$, and the system is assumed to be working if and only if edges which have not failed constitute at least one path from s to t . By *path*, we mean a simple path; that is, a chain of edges directed from s to t that involves no cycle; in the reliability literature such a path is usually called a *minimal path*. (For simplicity, let us assume graph vertices are not subject to failure, though the concepts stated here apply equally well to graphs where vertices also are "imperfect.") In this case the system would be considered working if and only if there were at least one simple path from s to t all of whose edges and vertices were working.)

A central point of Reference [1] is that if the inclusion-exclusion formula is utilized to determine the probability of all edges working in at least one $s - t$ path, terms in the formula may be associated with particular subgraphs of G . More specifically, a term is the probability that all edges are working in some set of paths, and these edges determine a subgraph of G . Other terms may refer to the same group of edges and hence to the same subgraph. The idea is to collect terms common to each subgraph and express the probability of the system working as

$$(1) \quad \sum d_H \Pr \{H\} ,$$

where the sum ranges over all subgraphs H having edge sets that can be expressed as a union of $s - t$ paths of G . $\Pr \{H\}$ is simply the probability that all edges of H are working. The coefficient d_H is an integer, which is easily shown to be the difference between the number of ways of representing the edge set of H as the union of an odd number of $s - t$ paths of G , and the number of ways of representing this set as a union of an even number of paths.

The ingenuity of the approach is due mainly to the fact that the coefficients d_H are either $+1$, -1 , or 0 . Satyanarayana and Prabhaker show that d_H is $+1$ or -1 for H an acyclic subgraph and give an algorithm for finding each of these subgraphs and its d_H value. They also offer a proof that $d_H = 0$ for H a cyclic subgraph, so cyclic subgraphs may be ignored.

The proof that $d_H = 0$ for the cyclic subgraphs seems to involve some technical points that require further clarification. In the present discussion, we establish this fact by a somewhat different argument, an argument that will hopefully be of some interest to the reader in its own right.

Most of the terminology used here is the same as that of Reference [1]. The subgraphs of G that are of interest are *p-graphs*; a *p-graph* is a directed graph with source s and sink t in which every edge is in at least one $s - t$ path. Given a set P of paths with initial vertex s and terminal vertex t , let $\text{Gr}(P)$ be the graph consisting precisely of the vertices and edges of paths in P . $\text{Gr}(P)$ is then a *p-graph*, and the set P is then called a *formation* of $\text{Gr}(P)$. P is an *odd formation*

of $Gr(P)$ if the set P contains an odd number of paths and an *even* formation if P contains an even number of paths.

If H is a p-graph, then $d_H = 0$ in formula (1) when the number of odd formations of H is equal to the number of even formations. Our argument that these numbers are equal for a cyclic p-graph involves two lemmas and a theorem. Lemma 2 and the theorem give the main ideas; Lemma 1 provides a technical fact used in proving Lemma 2. A number of examples are included to illustrate the various concepts.

Lemma 1:

Let G be a directed graph and P_{wz} be the set of all paths from vertex w to vertex z in G . If E is the set of edges common to every path in P_{wz} and if P_{wz} contains at least two paths, then there are at least two paths in P_{wz} whose only common edges are those in E .

Examples:

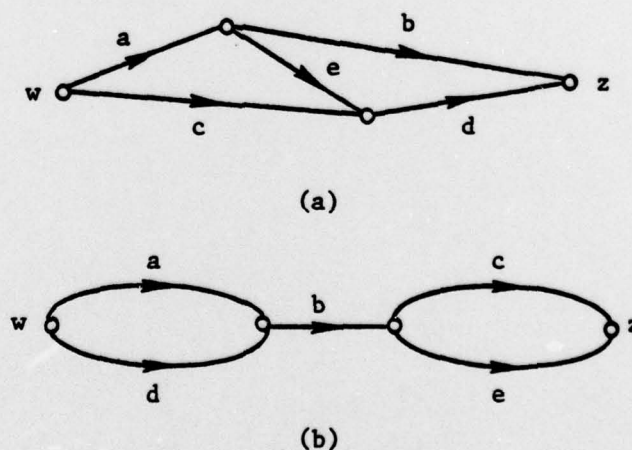


FIGURE 1

In Figure 1 (a), $E = \emptyset$ and paths $\{ab\}$ and $\{cd\}$ are edge disjoint. In Figure 1 (b), $E = \{b\}$ and paths $\{abc\}$ and $\{dbc\}$ have only edge b in common.

Proof:

If P_{wz} has at least two paths, it is not difficult to see that no path in P_{wz} can consist only of edges in E . Now consider the graph $Gr(P_{wz})$ as a flow network with source w and sink z . Let all edges in the set E have flow capacity 2 and all other edges of $Gr(P_{wz})$ have capacity 1. Every $w - z$ cut for the network has capacity at least 2, since no edges other than those in E are common to all paths. However, removal from $Gr(P_{wz})$ of any edge in E interrupts all paths from w to z , so the minimal $w - z$ cut has capacity 2. Thus, there exists an integer max flow having value 2. (See Reference [2].) No path in P_{wz} consists only of edges in E , so this flow may be decomposed into two flows, each having value 1 along a distinct path in P_{wz} . Because edges of these paths which are not in E have capacity 1, these two paths are edge disjoint except for edges in E . ||

Given a cyclic p-graph G and a simple cycle C in G , let M^C be a minimal set of paths in G , such that the p-subgraph $Gr(M^C)$ of G contains all edges of C . Let us call M^C a *minimal cover* for C . Evidently, there may be more than one minimal cover for C , so let $\{M_i^C, i = 1, \dots, k\}$ be the family of all minimal covers for C . By the minimality requirement, it is clear that $M_i^C \not\subseteq M_j^C$ for $i \neq j$.

Example:

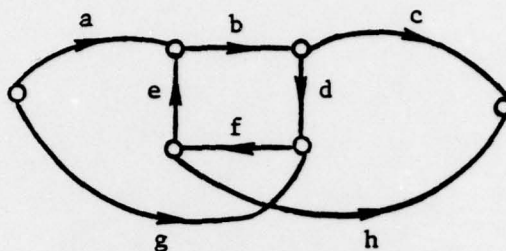


FIGURE 2

The cycle $C = \{bdfe\}$ has but one minimal cover, $M^C = \{\{abdfh\}, \{gfebc\}\}$. Note that paths $\{abc\}$ and $\{gfh\}$ each contain an edge of the cycle but are not in M^C .

Lemma 2:

Let G be a cyclic p-graph. Then G contains a simple cycle C and an $s - t$ path P^C such that P^C is not in any minimal cover for C . (The statement of the lemma is trivial when there is a path in G which contains no edge of some cycle. However, as the previous example illustrates, for some cyclic p-graphs, every path might intersect every cycle.)

Proof:

We might as well assume every $s - t$ path of G has an edge in common with every cycle of G . Suppose P is an $s - t$ path and let w be the first vertex along P which is contained in some (simple) cycle, say C' , and let ω be the edge of C' entering node w . Note that $w \neq s$, since otherwise edge ω enters the source, which is not valid for G a p-graph. Thus there is a subpath P_{sw} of P directed from vertex s to vertex w . The p-graph property leads us to conclude also that $w \neq t$, so the set P_{wt} of all subpaths from vertex w to the sink t is not empty. But no path in P_{wt} can involve a vertex of

the path P_{sw} other than w , since otherwise w would not then be the first cycle vertex in the original path P . Each combination of P_{sw} with a subpath in P_{wt} is thus a simple $s - t$ path. Figure 3 roughly illustrates the situation.

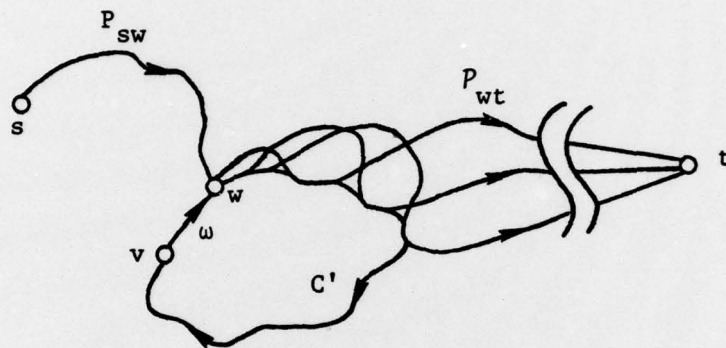


FIGURE 3

Under the initial assumption that every $s - t$ path of G has an edge in common with every cycle of G , it turns out that P_{wt} must contain at least two paths. The validity of the lemma does not depend on this fact, but our proof may be simplified by ignoring the case when P_{wt} contains a single path. That P_{wt} contains at least two paths is not difficult to show, though we do not show it here.

Let E be the set of edges common to every path in P_{wt} . By Lemma 1 above, there are at least two paths P'_{wt} and P''_{wt} whose only common edges are those in E . Since we have assumed that every $s - t$ path in G contains an edge of the cycle C' , every path in P_{wt} must pass through a vertex of C' . Let x be the last vertex along C' from vertex w , such that x is a vertex along path P'_{wt} or P''_{wt} . Without loss of generality, suppose path P'_{wt} passes through vertex x .

Consider the cycle C , formed from cycle C' by replacing the cycle path from vertex w to vertex x in C' with the subpath of P'_{wt} from w to x . The cycle C passes through each of its vertices but once, so C is, in fact, a simple cycle. (C could be the same as C' .) For convenience (or maybe inconvenience), Figure 4 attempts to illustrate the situation.

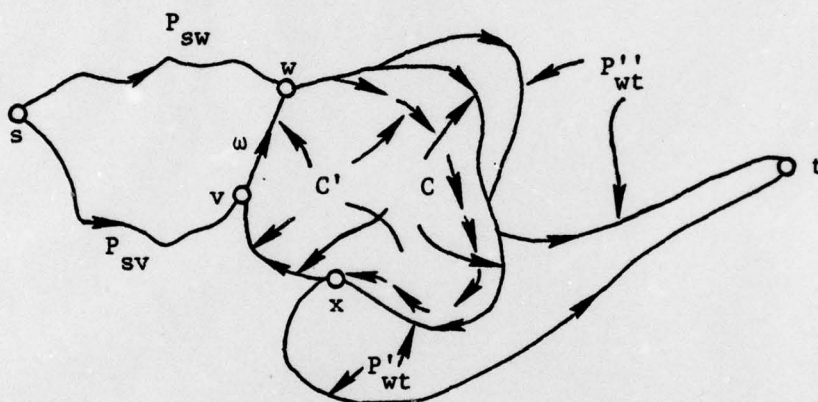


FIGURE 4

Finally, let $P^C = P_{sw} P'_{wt}$, the $s - t$ path formed by joining subpaths P_{sw} and P'_{wt} at vertex w . We maintain that P^C is not in any minimal cover M_1^C for cycle C . To see this, let M_1^C be any minimal cover of C and suppose P^ω is a path in M_1^C that contains edge ω . If v is the initial vertex of ω , P^ω may be expressed as $P_{sv} \omega P_{wt}$, where P_{sv} is some path from s to v and $P_{wt} \in P_{wt}$. Now note that the set of cycle C edges in P'_{wt} is a subset of the set E of edges common to all paths in P_{wt} . P_{wt} thus contains at least the edges of cycle C in P'_{wt} . Hence, P^C cannot be in M_1^C , since P^ω contains at least one more edge of C than P^C (namely, ω). Since M_1^C was an arbitrary cover of C , the lemma follows. ||

To further clarify the construction of Lemma 2, let us consider the following example:

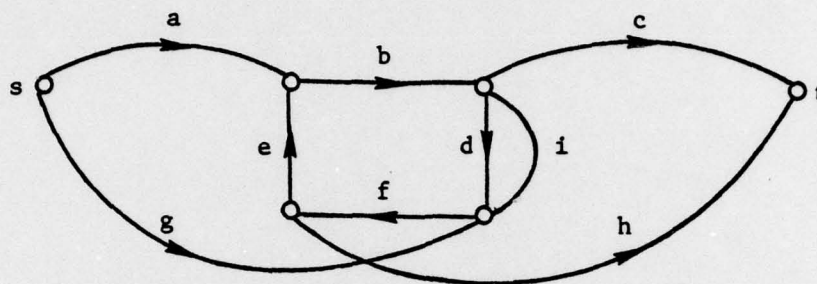


FIGURE 5

Take P_{sw} to consist of edge a , and $\{ebdf\}$ to be the cycle C' . The edge entering vertex w is then $\omega = e$. The set of paths P_{wt} is $\{\{bc\}, \{bdfh\}, \{bifh\}\}$. We can let $P'_{wt} = \{bc\}$ and P'_{wt} be either $\{bdfh\}$ or $\{bifh\}$. Suppose $P'_{wt} = \{bifh\}$. In this case, vertex x is the initial vertex of edge ω , i.e., $x = v$ in the vertex labeling of the proof. The new cycle C will then be $\{ebif\}$. There is only a single minimal cover of C , namely, $M^C = \{\{abifh\}, \{gfebc\}\}$. The path $P^C = P_{sw} P'_{wt} = \{abc\}$ contains only the cycle C edge b ; whereas, $P^\omega = \{gfebc\}$ contains not only cycle edge b , but also edge e .

Let \mathcal{P}_G be the family of all formations of a p-graph G and let P be any $s - t$ path in G . We partition the formations of \mathcal{P}_G into families $R(P)$ and $\{D_H(P) \mid H \text{ a proper p-subgraph of } G\}$ as follows: A particular formation $P \in \mathcal{P}_G$ is assigned to $R(P)$ if either (1) P does not contain path P , or (2) P contains path P but $P - \{P\}$ is also a formation of G . If P cannot be assigned to $R(P)$

then $H = Gr(P - \{P\})$ is a proper subgraph of G and P is assigned to $D_H(P)$. Note that the number of odd formations in $R(P)$ equals the number of even formations. As a rule, most of the families $\{D_H(P) \mid H \text{ a proper } p\text{-subgraph of } G\}$ will be empty. In some cases, $R(P)$ may be empty.

If $D_H(P)$ is nonempty, let $P_H = \{P - \{P\} \mid P \in D_H(P)\}$. P_H consists of all formations of the p -graph H , since if Q is any formation of H , $Q \cup \{P\}$ must be a formation of G ; hence, $Q \cup \{P\} \in D_H(P)$ and $Q \in P_H$.

As an example of this partition, consider the graph of Figure 6.

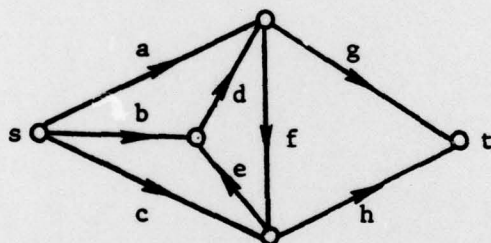


FIGURE 6

The paths are:

$$\begin{array}{ll}
 p^1 = \{ag\} & p^4 = \{bdfh\} \\
 p^2 = \{afh\} & p^5 = \{cedg\} \\
 p^3 = \{bdg\} & p^6 = \{ch\} .
 \end{array}$$

The set P_G of formations of G is

$$p^1 = \{p^2 p^3 p^5\}$$

$$p^5 = \{p^1 p^2 p^4 p^5\}$$

$$p^2 = \{p^1 p^2 p^3 p^5\}$$

$$p^6 = \{p^1 p^3 p^4 p^5\}$$

$$p^3 = \{p^1 p^4 p^5\}$$

$$p^7 = \{p^2 p^3 p^4 p^5\}$$

$$p^4 = \{p^2 p^4 p^5\}$$

$$p^8 = \{p^1 p^2 p^3 p^4 p^5\}$$

$$p^9 = \{p^2 p^3 p^5 p^6\}$$

$$p^{13} = \{p^1 p^2 p^4 p^5 p^6\}$$

$$p^{10} = \{p^1 p^2 p^3 p^5 p^6\}$$

$$p^{14} = \{p^1 p^3 p^4 p^5 p^6\}$$

$$p^{11} = \{p^1 p^4 p^5 p^6\}$$

$$p^{15} = \{p^2 p^3 p^4 p^5 p^6\}$$

$$p^{12} = \{p^2 p^4 p^5 p^6\}$$

$$p^{16} = \{p^1 p^2 p^3 p^4 p^5 p^6\} .$$

Let us choose p^3 and partition according to this path:

$$R(p^3) = \left\{ \begin{array}{l} p^3, p^4, p^5, p^{11}, p^{12}, p^{13} \\ p^6, p^7, p^8, p^{14}, p^{15}, p^{16} \end{array} \right\} .$$

Note the first 8 formations in $R(p^3)$ do not contain p^3 , but the last 8 do. Finally, the only proper p -subgraph H for which $D_H(p^3)$ is nonempty is that of Figure 7; for this subgraph, $D_H(p^3) = \{p^1, p^2, p^9, p^{10}\}$.

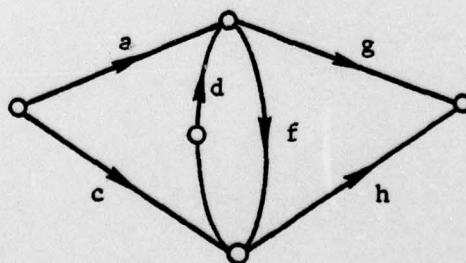


FIGURE 7

Theorem:

Let G be a cyclic p -graph. The number of odd formations of G is equal to the number of even formations.

Proof:

We will use induction on the number of paths of G .

It is easy to verify that there are no cyclic p -graphs having less than 4 paths, so let G have $n = 4$ paths. By Lemma 2, G has a cycle C and a path P^C that is contained in no minimal cover M_1^C for that cycle. Partition the family P_G of formations as indicated above, using the path P^C . We maintain that $D_H(P^C)$ is empty for every p -subgraph H of G . To see this, note that if $P \in D_H(P^C)$, there is at least one minimal cover M_1^C in $P - \{P^C\}$, since P^C is in no M_1^C . But then $H = \text{Gr}(P - \{P^C\})$ is cyclic, which is not possible because H has 3 or less paths. Thus $P_G = R(P^C)$, and since the number of even formations of G in $R(P^C)$ is equal to the number of odd formations, the theorem is true when G has 4 paths.

Now suppose the theorem is true for all cyclic p -graphs having m paths, $4 \leq m \leq n - 1$. Let G be a cyclic p -graph having n paths. By Lemma 2 again, G has a cycle C and a path P^C that is in no minimal cover for C . Partitioning the family P_G of all formations of G , we get families $R(P^C)$ and $\{D_H(P^C) \mid H \text{ a proper } p\text{-subgraph of } G\}$. If $D_H(P^C)$ is empty for all p -subgraphs H of G , then $P_G = R(P^C)$ as above, and the number of even formations of G equals the number of odd formations. Hence, suppose $D_H(P^C)$ is nonempty for some H . Since P^C is contained in no M_1^C , H must be cyclic and H contains $n - 1$ or less paths. By the induction assumption, the family P_H of all

formations of H has the same number of even as odd formations. Each formation of G in $\mathcal{D}_H(P^C)$ can be produced by appending P^C to one of the formations in \mathcal{P}_H . Hence, for every nonempty family $\mathcal{D}_H(P^C)$, the number of even formations in $\mathcal{D}_H(P^C)$ equals the number of odd formations. Thus, each family in the partition of \mathcal{P}_G is either empty or contains the same number of even as odd formations, and the theorem follows. ||

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- [1] Satyanarayana, A. and A. Prabhakar, "New Topological Formula and Rapid Algorithm for Reliability Analysis of Complex Networks," IEEE Transactions on Reliability, Vol. R-27, No. 2, (June 1978).
- [2] Ford, L. R., Jr. and D. R. Fulkerson, FLOWS IN NETWORKS, Princeton University Press, Princeton, New Jersey, (1962).